Problem Set 2 due March 4, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

(1) Prove, by direct computation, that if:

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then:

$$X^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The number $ad - bc$ must be invertible (i.e. non-zero) for X^{-1} to make sense. *(10 points)*

(2) Assume A, B, C, D are all 2×2 matrices, and assume that:

$$AD - BC = DA - CB = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad \text{and} \qquad AB = BA \qquad \text{and} \qquad CD = DC$$

for some number $\lambda \neq 0$. Under these assumptions, what is the inverse of the block 4×4 matrix:

$$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Prove your formula for X^{-1} by showing all the steps. *(10 points)*

Problem 2:

(1) Compute the inverses of the matrices:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix}$$

for various numbers a, b, c, a', b', c'. **You must use the Gauss-Jordan elimination procedure** outlined on page 13 of the Lecture notes (or page 86 of the textbook), that is, by starting from the augmented matrices $\left[\ L \mid I\ \right]$ and $\left[\ U \mid I\ \right]$. *(10 points)*

(2) Compute the inverse of the matrix:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

for any three non-zero numbers d_1, d_2, d_3. *(5 points)*

(3) Consider the 3×3 matrix $A = LDU$, where L, D and U are as above. Write A^{-1} as a product of three matrices, and as a single matrix. Note that the formula for A^{-1} as a single matrix will not be particularly pretty, but it will give you some practice with matrix multiplication. *(10 points)*

Problem 3:

(1) Gaussian elimination involves doing row operations on matrices. But what if instead one did column operations on a matrix? For example, start with:

$$A = \begin{bmatrix} \boxed{-1} & -3 & 2 \\ 2 & \boxed{\boxed{7}} & 5 \\ -1 & 0 & 3 \end{bmatrix}$$

and add appropriate multiples of the first column to the second and third columns, so that all entries to the right of the box vanish. Then add an appropriate multiple of the second column to the third column, so that all entries to the right of the double box vanish. Carry out this process by showing all the steps. *(10 points)*

(2) Describe each of the steps in the process above as multiplying A on the right with an appropriate matrix. What factorization of A does this produce? *(10 points)*

Problem 4:

(1) Consider the matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 4 & 7 & 6 \end{bmatrix}$$

and write it as the sum of a symmetric matrix and an anti-symmetric matrix (recall that while a symmetric matrix is one such that $S = S^T$, an anti-symmetric matrix is one such that $A = -A^T$). *(10 points)*

(2) For a general matrix X, suppose you want to write it as $X = S + A$, where S is symmetric and A is anti-symmetric. Can you find formulas for S and A in terms of X only? *(5 points)*

Problem 5:

(1) Consider the matrix:

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 7 & 6 & 0 \\ 0 & 4 & 15 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix}$$

and compute its LU factorization. *(10 points)*

(2) Based on part (1), how do you think the LU factorization of a matrix of the form:

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & a_5 & b_5 & c_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 & b_6 & c_6 \\ 0 & 0 & 0 & 0 & 0 & a_7 & b_7 \end{bmatrix}$$

will look like? (Just a general guess on how the matrices L and U will look will suffice for now. Hint: L and U will have a lot of zeroes. Where do you think they are located?) *(5 points)*

(3) In the generality of part (2), what are the entries of U above the diagonal, and why? *(5 points)*